6th homework assignment **Dynamical Systems** **SS 10**
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Problem 21. **points**

Consider the initial-value problem

$$\dot{x} = f(x) := x, \qquad x(0) = x_0 := 1$$

on the time interval $0 \leq t \leq T$ *for fixed* $T > 0$. *Calculate an approximate solution* $x(T)$ *analytically*

(i) by Picard iteration, i.e. determine the n-th Picard iterate $x^{[n]}(T)$;

$$x^{[k+1]}(t) = x_0 + \int_0^t f(x^{[k]}(\tau))\mathrm{d}\tau, \quad x^{[0]}(t) \equiv x_0$$

(ii) by explicit Euler scheme, i.e. determine the value $x^{[n]}$ *after n Euler steps of stepsize* $h^{[n]} = \frac{T}{n}$,

$$x_{k+1}^{[n]} = x_k^{[n]} + h^{[n]} f\left(x_k^{[n]}\right), \quad x_0^{[n]} = x_0$$

Compare!

We have

$$\dot{x} = x, \qquad x(0) = 1$$

so the solution is

$$x(t) = e^t$$

(i)

$$\begin{aligned}
& x^{[0]}(t) \equiv 1 \\
\Rightarrow \quad & x^{[1]}(t) = 1 + \int_0^t x^{[0]}(\tau)\mathrm{d}\tau = 1 + t \\
\Rightarrow \quad & \dots \\
\Rightarrow \quad & x^{[n]}(T) = 1 + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^2 + \cdots + \frac{1}{n!}T^n \\
\Rightarrow \quad & \lim_{\mathrm{n}\to\infty} x^{[n]}(T) = \sum_{n=0}^{\infty} \frac{1}{n!}T^n =: \exp(T) \qquad \text{(Taylor series)}
\end{aligned}$$

(ii)

$$\begin{aligned}
& x_{k+1}^{[n]} = x_k^{[n]} + h^{[n]} x_k^{[n]} = \left(1 + h^{[n]}\right) x_k^{[n]} \\
\Rightarrow\quad & x_n^{[n]} = \left(1 + h^{[n]}\right)^n \cdot 1 = \left(1 + \frac{T}{n}\right)^n \\
\Rightarrow\quad & \lim_{n\to\infty} x_n^{[n]} = \lim_{n\to\infty} \left(1 + \frac{T}{n}\right)^n =: \exp(T) \qquad \text{(another definition of exp)}
\end{aligned}$$

So both approximate solutions converge to the correct solution.
The Picard iteration converges faster:
For the $(k+1)$-th step there is only $\frac{T^n}{n!}$ added to the previous value $x^[k]$, which (even for big T) becomes small very fast.
$x_n^{[n]}$ is easier to calculate (only multiplication, no integrals) but converges slower.

Problem 22. **points**

The initial-value problem

$$\dot{x} = f(x) := x^2, \qquad x(0) = x_0 := 1$$

has a solution for $-\infty < t < 1$ *with "blow-up",* $\lim_{t\to 1} x(t) = +\infty$.
Let $(xk)_{k\in\mathbb{N}}$ *be the series of Picard iterates:*

$$\begin{aligned}
x_0(t) &\equiv x_0 \\
x_{n+1}(t) &= x_0 + \int_0^t f(x_n(s))\mathrm{d}s
\end{aligned}$$

(i) Prove: $x_k(t)$ *is defined for all* $k \in \mathbb{N}$ *and* $t \in \mathbb{R}$.

(ii) Calculate $x_1(t), x_2(t), x_3(t)$ *and* $x_4(t)$ *explicitly.*

(iii) Determine all $t \geq 0$ *such that* $x_k(t)$ *converges to the solution* $x(t)$ *of the initial-value problem, as* $k \to \infty$.

(i) **Proof:** Polynomes are defined on all of $\mathbb{R}$

$$\begin{aligned}
x_0 &\equiv 1 \quad \in \text{polynomes on } \mathbb{R} \\
x_{k+1}(t) &= 1 + \int_0^t \underbrace{(x_k(s))^2}_{\text{polynome}} \mathrm{d}s \\
&= 1 + \underbrace{\int_0^t p(s)\mathrm{d}s}_{\text{integral is polynome again}} \\
&=: P(t) \quad \in \text{polynomes on } \mathbb{R}
\end{aligned}$$

So by induction $x_k(t)$ is defined $\forall t \in \mathbb{R}$. □

Prove: $x_k(t)$ is defined for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$.

(ii)

$$
\begin{aligned}
x_1(t) &= 1+t \\
x_2(t) &= 1+\int_0^t (1+s)^2 \mathrm{d}s = 1+t+t^2+\frac{1}{3}t^3 \\
x_3(t) &= 1+\int_0^t (1+s+s^2+\frac{1}{3}s^3)^2 \mathrm{d}s \\
&= 1+t+t^2+t^3+\frac{2}{3}t^4+\frac{1}{3}t^5+\frac{1}{9}t^6+\frac{1}{63}t^7 \\
x_4(t) &= 1+\int_0^t (t+t^2+t^3+\frac{2}{3}t^4+\frac{1}{3}t^5+\frac{1}{9}t^6+\frac{1}{63}t^7)^2 \mathrm{d}s \\
&= 1+\int_0^t \Bigg(1+2t+3t^2+4t^3+\frac{13}{3}t^4+4t^5+\frac{29}{9}t^6+\frac{142}{63}t^7+\frac{86}{63}t^8 \\
&\qquad \frac{44}{63}t^9+\frac{55}{189}t^{10}+\frac{2}{21}t^{11}+\frac{13}{567}t^{12}+\frac{2}{567}t^{13}+\frac{1}{3969}t^{14}\Bigg)\ \mathrm{d}s \\
&= 1+t+t^2+t^3+t^4+\frac{13}{15}t^5+\frac{2}{3}t^6+\frac{29}{63}t^7+\frac{71}{252}t^8+\frac{86}{567}t^9 \\
&\quad +\frac{22}{315}t^{10}+\frac{5}{189}t^{11}+\frac{1}{126}t^{12}+\frac{1}{567}t^{13}+\frac{1}{3969}t^{14}+\frac{1}{59535}t^{15}
\end{aligned}
$$

(iii) By the results of (ii) we assume, that $x_n(t)$ are in the form

$$x_n(t) = 1+t+t^2+t^3+\cdots+t^n+\text{higher terms} \tag{1}$$

We also see (because this is quite a simple ODE) that the exact solution is

$$x(t) = -\frac{1}{t-1}$$

which can be expressed as a Taylor series for $t \in [0,1)$ which is the time where a solution to the ODE exists

$$x(t) = \sum_n^\infty t^n$$

This would mean convergence of the approximate solution for $n \to \infty$ everywhere ($\forall t \in [0,1)$) So we need to proof (1)

Proof: By induction over n we get

$$
\begin{aligned}
x_0 &= 1 && \text{is in the correct form} \\
x_{n+1} &= 1+\int_0^t (x_n(s))^2 \mathrm{d}s
\end{aligned}
$$

We are only interested in terms $a_k t^k$ with $k \leq n+1$. So if we square the terms of x_n, which supposedly is in the form (1), only the terms with $k \leq n$ can support to those because for $k > n \Rightarrow \int a_k t^k = \frac{a_k}{k} t^{k+1}$ which isn't of any interest.

$$\begin{aligned}
\Rightarrow x_{n+1} &= 1 + \int_0^t (1 + s + s^2 + \cdots + s^n)^2 \mathrm{d}s + \text{higher terms} \\
&= 1 + \int_0^t (1 + 2s + 3s^2 + \cdots + (n+1)s^n) \mathrm{d}s + \text{higher terms} \\
&= 1 + (t + t + t^2 + \cdots + t^{n+1}) + \text{higher terms}
\end{aligned}$$

This is in the correct form, as supposed.
So the iteration converges to the Taylor series of the correct solution, because the *'higher terms'* become small for $t < 1$.

□